Algebraic Reasoning About Timeliness

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Introduction
Why predict performance?

* Weather forecast of today can’t arrive tomorrow!
* Without performance prediction
  * Performance issues exposed late in design cycle
  * Either:
    * Re-architect the design, with cost and delay, or
    * Allocate excessive resources, with cost and inefficiency.
* With performance prediction
  * Performance issues exposed early in design cycle
  * Re-architect the design before time and money spent, and
  * Control resources, avoiding cost and inefficiency.

See [1, §1.1] for more:

* *Mind Your Outcomes: The \( \Delta QSD \) Paradigm for Quality-Centric Systems Development and Its Application to a Blockchain Case Study.* Computers 11(3): 45 (2022)

Why does IOG fund research on performance?

- Good Starting Point: Kevin Hammond’s Keynote in Lambda Days 2023 https://tinyurl.com/3t42t3wn
- IOG is a prominent blockchain company. https://iohk.io
- The effective operation of the Cardano network depends on a performance aware design.
- The ΔQSD Team on Formalising Performance Aspects
Last Year’s DisCoTec Tutorial by Peter VAN ROY

The $\Delta$QSD Paradigm for System Development

June 13, 2022
DisCoTec Tutorial

Peter Van Roy
Université catholique de Louvain

Neil Davies, Peter Thompson
Predictable Network Solutions Ltd.

Seyed Hossein Haeri
PLWorkz

https://www.youtube.com/watch?v=iBYZEJZwKm0
What's timeliness?

Timeliness is delivering results within the required time bounds (sufficiently often).
Cache Example

* Outcome Diagrams
* Outcome Expressions
* An Algebraic Perspective on Timeliness
* Where is the algebra?
Cache Example

- Outcome Diagrams
- Outcome Expressions
- An Algebraic Perspective on Timeliness
- Where is the algebra?
Big Picture

Block Diagram

Cache Example
Hit or Miss

Note:
- Outcomes: What the System Gains by Performing One of its Tasks
  - NOT System States
  - NOT Subsystems
  - NOT Classes/Objects
- Probabilistic Choice (↔)
Lookup from Main Memory

Note:
* Sequential Composition
* Left-to-Right Causality
Error Correction

- Cache Example

- Algebraic Results

- Q&A

Diagram:

- c-miss
- c-hit
- ECC fail
- main

Probabilities:

- 95%
- 5%
- $10^{-16}$
Timeout (1 of 3)

Time-Bounded Network Connection Back & Forth
Timeout (2 of 3)

- Cache Example
- Algebraic Results
- Q&A

![Diagram of cache example with nodes and edges]
Note:

* Any-to-Finish (∃)
Cache Example

- Outcome Diagrams
- Outcome Expressions
- An Algebraic Perspective on Timeliness
- Where is the algebra?
Expression:

\[
\text{main} \xleftrightarrow{1} 10^{-16} \bot
\]

Note:
* “\(\bot\)” is for failure.
Expression:

\[\text{net} \rightarrow\rightarrow (\text{main} \xrightarrow{1} 10^{-16} \perp) \rightarrow\rightarrow \text{net}\]

Note:

* “\(\rightarrow\rightarrow\)” is for sequential composition.
Expression:

\[(\text{net} \\xrightarrow{} \text{main} \xleftrightarrow{1} \text{main} \xrightarrow{10^{-16}} \text{net}) \parallel \exists \text{t-out})\]

Note:

* “\(\parallel \exists\)” is for any-to-finish.
\[
c\text{-hit} \quad [95\%] \Rightarrow (\text{c-miss} \quad \text{miss} \quad \text{net}\quad \text{mread}\quad 1\quad \text{main}\quad \text{mreturn}\quad \text{net} \quad \text{hit} \quad \text{miss}) \quad || \exists \quad \text{t-out})
\]
Cache Example

- Outcome Diagrams
- Outcome Expressions
- An Algebraic Perspective on Timeliness
- Where is the algebra?
What’s a $\Delta Q$?

- Quality Attenuation
- A Measure for Delay (and Failure)
- Represented using a Cumulative Distribution Function (CDF)
- Improper Random Variable (IRV) [2]
Timeliness Semantics

Definition (Haeri et al. [1]): Given a basic assignment $\Delta_\circ[.] : \mathcal{B} \rightarrow \Delta$, define $\Delta Q[.]_{\Delta_\circ} : \mathcal{O} \rightarrow \mathcal{I}$ such that...
Definition (Haeri et al. [1]): Given a basic assignment \( \Delta_\circ[\cdot] : \mathbb{B} \to \Delta \), define \( \Delta_\circ \square[\cdot]_\circ : \square \to \square \) such that

\[
\Delta_\circ \square[\cdot]_\circ = \begin{cases} 
1 & \text{when } \Delta_\circ[\cdot] \notin \square \\
\Delta_\circ[\cdot]_\circ & \text{otherwise}
\end{cases}
\]

\[
\Delta_\circ \bigcirc[\circ \rightarrow \circ']_\circ = \Delta_\circ \bigcirc[\circ]_\circ \times \Delta_\circ \bigcirc[\circ']_\circ
\]

\[
\Delta_\circ \bigcirc[\circ \rightleftharpoons \circ']_\circ = \frac{m}{m+m'} \Delta_\circ \bigcirc[\circ]_\circ + \frac{m'}{m+m'} \Delta_\circ \bigcirc[\circ']_\circ
\]

\[
\Delta_\circ \bigcirc[\circ \leftarrow \circ']_\circ = \Delta_\circ \bigcirc[\circ]_\circ \times \Delta_\circ \bigcirc[\circ']_\circ
\]

\[
\Delta_\circ \bigcirc[\circ \leftrightarrow \circ']_\circ = \Delta_\circ \bigcirc[\circ]_\circ + \Delta_\circ \bigcirc[\circ']_\circ - \Delta_\circ \bigcirc[\circ]_\circ \times \Delta_\circ \bigcirc[\circ']_\circ
\]
Definition (Haeri et al. [1]): Given a basic assignment $\Delta_\circ[.] : \mathbb{B} \rightarrow \Delta$, define $\Delta Q[.]_\circ : \mathbb{O} \rightarrow \mathbb{I}$ such that

$$
\Delta Q[[\beta]]_\circ. = \begin{cases} 
1 & \text{when } \Delta_\circ[[\beta]] \notin \mathbb{I} \\
\Delta_\circ[[\beta]] & \text{otherwise}
\end{cases}
$$

$$
\Delta Q[[o \rightarrow o']]_\circ. = \Delta Q[[o]]_\circ. \ast \Delta Q[[o']]_\circ.
$$

$$
\Delta Q[[o \rightarrow m o']]_\circ. = \frac{m}{m+m'} \Delta Q[[o]]_\circ. + \frac{m'}{m+m'} \Delta Q[[o']]_\circ.
$$

$$
\Delta Q[[o \parallel o']]_\circ. = \Delta Q[[o]]_\circ. \times \Delta Q[[o']]_\circ.
$$

$$
\Delta Q[[o \parallel o']]_\circ. = \Delta Q[[o]]_\circ. + \Delta Q[[o']]_\circ. - \Delta Q[[o]]_\circ. \times \Delta Q[[o']]_\circ.
$$
» $\Delta Q$ of the Cache Example

Given

$$\Delta \supseteq \{\Delta Q_{c\text{-hit}}, \Delta Q_{c\text{-miss}}, \Delta Q_{mem}, \Delta Q_{t\text{-out}}, \Delta Q_{mem}, \Delta Q_{t\text{-out}}\},$$

one calculates:

$$\Delta Q_{obs} = 0.95 \times \Delta Q_{c\text{-hit}} + 0.05 \times (\Delta Q_{c\text{-miss}} \times (\Delta Q_{mem} + \Delta Q_{t\text{-out}} - \Delta Q_{mem} \times \Delta Q_{t\text{-out}})),$$

where

$$\Delta Q_{mem} = \Delta Q_{\text{net}} \times (1 - 10^{-16}) \times \Delta Q_{\text{main}} \times \Delta Q_{\text{net}}.$$
Timeliness for the Cache

Δ\text{Q}_\text{req}:

* 10% of queries up to 4ms
* 50% of queries up to 6ms
* 90% of queries up to 14ms
* 10% of queries never
Cache Example

- Outcome Diagrams
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- Where is the algebra?
Expression:

\[ c\text{-hit}^{[95\%]} \leftrightarrow (c\text{-miss} \rightarrow \text{main}^{1 \over 10^{-16}} \perp) \]
Algebraic Manipulation

\[
c \quad \text{hit} \ [95\%] \quad (c \quad \text{miss} \quad \rightarrow \quad \text{main} \quad \frac{1}{10^{-16}} \quad \bot) \\
\]

\[
c \quad \text{hit} \ [95\%] \quad ((c \quad \text{miss} \quad \rightarrow \quad \text{main}) \quad \frac{1}{10^{-16}} \quad \bot) \\
\]

\[
(c \quad \text{hit} \quad [\cdot] \quad (c \quad \text{miss} \quad \rightarrow \quad \text{main})) \quad [q] \quad \bot
\]

where \( q = (1 - 0.05 \times 10^{-16}) = 0.999999999999999995 \).

\* 17 nines vs 9 nines of Ericsson AXD301

Not a Guarantee for Success!

Just ruling out infeasibility with this level of information.
Benefit of Algebraic Manipulation

$q = (1 - 0.05 \times 10^{-16}) = 0.999999999999999995$

* What if we had already implemented the cache?
* Will simply throwing more hardware at it work?
* Re-architecture from scratch?
Algebraic Results
## Algebraic Structures

<table>
<thead>
<tr>
<th>O with</th>
<th>Forms</th>
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<tbody>
<tr>
<td>⇄</td>
<td>magma</td>
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<tr>
<td>⊏</td>
<td>commutative monoid with (\top) and (\bot) as the identity and absorbing elements</td>
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<tr>
<td>⊏∧</td>
<td>commutative monoid with (\top) and (\bot) as the identity and absorbing elements</td>
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<tr>
<td>⊏∃</td>
<td>commutative monoid with (\bot) and (\top) as the identity and absorbing elements</td>
</tr>
</tbody>
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**Note:** Neither \(\mathbin{\|\wedge}\) nor \(\mathbin{\|\exists}\) nor their combination form the familiar richer algebraic structures.
Equivalences Containing Constant Outcomes

\[
\begin{align*}
\bot & \equiv \bot = \bot \\
\top & \equiv \top = \top \\
\top & \equiv \forall \top = \top \\
\bot & \equiv \exists \bot = \bot \\
\top & \equiv \bot = \bot \\
\top & \equiv \bot = \bot \\
\bot & \equiv \top = \bot \\
\bot & \equiv \bot = \bot \\
\top & \equiv \forall \bot = \bot \\
\bot & \equiv \exists \top = \bot \\
[\rho] & \equiv [\sigma] \top = [\sigma(1-p)] \left[ \frac{\rho}{1-q(1-p)} \right] \top \\
\bot & \equiv [\rho] \left( \bot \equiv [\rho] \top \right) = \bot \left[ \rho + (1-p)q \right] \top
\end{align*}
\]
» Equivalences Containing Constant Outcomes

\[ \bot \iff \bot = \bot \]
\[ \top \iff \top = \top \]
\[ \bot \iff \bigcirc \rightarrow \bullet \bot = \bot \]
\[ \top \iff \bigcirc \rightarrow \bullet \top = \top \]
\[ \bot \iff \bigcirc \rightarrow \exists \bullet \top = \top \]

\[ o_1 \iff \bigcirc \rightarrow \bullet \bigcirc \iff \bot ) = ( o_1 \iff \bigcirc \rightarrow \bullet o_2 ) \iff \bot \]
\[ ( o_1 \iff \bot ) \iff \bigcirc \rightarrow \bullet o_2 = ( o_1 \iff \bigcirc \rightarrow \bullet o_2 ) \iff \bot \]
\[ ( o_1 \iff \top ) \iff \bigcirc \rightarrow \bullet o_2 = ( o_1 \iff \bigcirc \rightarrow \bullet o_2 ) \iff o_2 \]
\[ o_1 \iff \bigcirc \rightarrow \bullet ( o_2 \iff \top ) = ( o_1 \iff \bigcirc \rightarrow \bullet o_2 ) \iff o_1 \]

\[ o_1 \[ p \] \iff \bigcirc \rightarrow \bullet \bigcirc \iff \bot ) = ( o_2 \[ q \] \iff \top \]
\[ ( \bot \iff \bigcirc \rightarrow \bullet \top ) = ( \bot \iff \bigcirc \rightarrow \bullet o \) \iff \bot \[ p + (1-p)q] \iff \top \]

[26/32]


» Equivalences Containing Constant Outcomes

\[
\begin{align*}
\bot & \iff \bot = \bot \\
T & \iff T = T \\
\bot & \iff \bot = \bot \\
T & \iff T = T \\
\bot & \iff \bot = \bot \\
T & \iff T = T \\
\end{align*}
\]

\[
\begin{align*}
o_1 & \iff (o_2 \iff \bot) = (o_1 \iff o_2) \iff \bot \\
(o_1 \iff \bot) & \iff o_2 = (o_1 \iff o_2) \iff \bot \\
(o_1 \iff T) & \iff o_2 = (o_1 \iff o_2) \iff o_2 \\
o_1 & \iff (o_2 \iff T) = (o_1 \iff o_2) \iff o_1 \\
o_1 \left[\frac{p}{q}\right] (o_2 \left[\frac{q}{1-p}\right] T) & = o_2 \left[\frac{q(1-p)}{1-q(1-p)}\right] (o_1 \left[\frac{p}{1-q(1-p)}\right] T) \\
\bot \left[\frac{p}{q}\right] (\bot \left[\frac{q}{1-p}\right] o) & = \bot \left[\frac{p+(1-p)q}{1-p}\right] o \\
\end{align*}
\]

ECC followed by a net failure is as timely as failure itself!
Equivalences Containing Constant Outcomes

\[
\begin{align*}
\bot & \iff \bot = \bot \\
T & \implies o = o \\
T & = o \iff T = o \\
\top & \implies \exists o = o \\
\top & \iff \forall o = o
\end{align*}
\]

\[
\begin{align*}
o_1 \implies o_2 = \bot & \iff (o_1 \implies o_2) = \bot \\
(o_1 \iff \bot) & \implies o_2 = (o_1 \implies o_2) = \bot \\
(o_1 \iff \top) & \implies o_2 = (o_1 \implies o_2) = o_2 \\
o_1 \implies (o_2 = \top) & \iff (o_1 \implies o_2) = o_1 \\
o_1 \left[\frac{p}{p}\right] \iff (o_2 \left[\frac{q}{q}\right] \top) & \iff o_2 \left[\frac{q(1-p)}{1-q(1-p)}\right] (o_1 \left[\frac{p}{1-q(1-p)}\right] \top) \\
\bot \left[\frac{p}{p}\right] \iff (\bot \left[\frac{q}{q}\right] o) & \iff \bot \left[\frac{p+(1-p)q}{p+(1-p)q}\right] o
\end{align*}
\]

Seen at the Algebraic Manipulation of the Cache Example
Distributivity

Let $o_1, o_2, o_3 \in$ and $p \in \{\rightarrow, \forall, \exists\}$. Then,

- $\circ \circ \ time \models o_1 \ p \ (o_2 \leftrightarrow o_3) = (o_1 \ p \ o_2) \leftrightarrow (o_1 \ p \ o_3),$ 
  and

- $\circ \circ \ time \models (o_1 \leftrightarrow o_2) \ p \ o_3 = (o_1 \ p \ o_3) \leftrightarrow (o_2 \ p \ o_3).$

Bad News! Only 3 Out of the Possible 15
Summary

* Formalisation of $\Delta QSD$ – Ongoing Project
* Algebraic Manipulations $\Rightarrow$ Tool Support
* Properisation
  * Ordinary $\Delta Q[\ldots]$ doesn’t work!
  * The First IRV Body of Theorems Ever!
Questions?
» Thank you very much!